Available online @ https://jjem.jnnce.ac.in
https:www.doi.org/10.37314/JJEM.2021.050210
Indexed in International Scientific Indexing (ISI)
Impact factor: 1.395 for 2021-22
Published on: 31 January 2022

A Study on Main Scalar and Geodesic Equation of a Finsler Space with Special (α, β)-metric

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Abstract

In the present article, we find the condition for the Finsler space with special (α, β)-metric $L = \mu\left(\frac{\alpha^2}{\beta}\right) + \nu\left(\frac{\beta^2}{\alpha}\right)$; μ and ν are constants, to be of Berwald space. Further, we determine the main scalar of two dimensional Finsler space and finally we derive the geodesic equation for a Finsler space with above mentioned metric.

Keywords: *(α, β)-metrics, Main Scalar I, Geodesic equation, Finsler space.*

1. Introduction

The Finsler space $F^n = \left(M^n, L(\alpha, \beta)\right)$ is said to have an (α, β)-metric if L is positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. The interesting examples of an (α, β)-metric are Randers metric, Kropina metric and Matsumoto metric.

Kropina metrics are a kind of (α, β)-metric with $\phi(s) = \frac{1}{s}$ i.e., $F = \frac{\alpha^2}{\beta}$. Kropina metrics were first introduced by L. Berwald in connection with a two dimensional Finsler space with rectilinear extremals and were investigated by V. K. Kropina[5]. Although it has important applications in the general dynamic system represented by Lagrangian functions. And also there are many developments in the study of Kropina metrics. And also, we are considering another important kind of (α, β)-metric in the form of $F = \frac{\beta^2}{\alpha}$.

In this article, we have considered the special (α, β)-metric which is a linear combination of $L_1 = \frac{\alpha^2}{\beta}$ and $L_2 = \frac{\beta^2}{\alpha}$ in the form of

$$L = \mu\left(\frac{\alpha^2}{\beta}\right) + \nu\left(\frac{\beta^2}{\alpha}\right). \tag{1.1}$$

When $\mu = 0$, equation (1.1) is homothetic to $L = \frac{\alpha^2}{\beta}$. In [8], the authors found some results regarding the linear combinations of special (α, β)-metrics, such as Berwlad connections, difference vector, geodesic equation and main scalar in two-dimensional case.

In the present article, we are considering the linear combination of the Finsler metric in the form of (1.1) and find the condition for a Finsler space with the metric (1.1) to be of Berwald space. In the next part, we discuss the condition for main scalar for two dimensional Finsler space. Finally, we derive the geodesic equation for a Finsler space with the metric (1.1).

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Let T_xM be the tangent space at $x \in M$ and $TM = U_{x\in M} T_xM$ the tangent space of M. Each element of TM has of the form (x, y), where $x \in M$ and $y \in T_xM$.

Definition 2.1. A Finsler metric on a manifold M is a function $L: TM \to [0, \infty)$, having the following properties:

i) L is C^∞ on TM_0,
ii) $L(x, \lambda y) = \lambda L(x, y), \lambda > 0$,
iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $\frac{L^2}{2}$ given by
$$g_{ij}(x, y) = \left[\frac{1}{2}L^2\right] y^i y^j$$
is positive definite.

For a given Finsler metric $F = F(x, y)$
$$\frac{d^2x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0, \qquad (2.1)$$
where $G^i = G^i(x, y)$ is called the geodesic coefficient which is given by
$$G^i(x, y) = \frac{1}{4} g^{il}(x, y)\left[2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \right.$$
∂gjk∂xlx,yyjyk,

where the matrix g^{ij} means the inverse of matrix (g_{ij}).

For an (α, β)-metric, the space $R^n = (M^n, \alpha)$ is called the associated Riemannian space with $F^n = (M^n, L(\alpha, \beta))$ ([1], [6]). In the following discussion, the Riemannian metric α is not supported to be positive definite and we shall restrict our discussion to a domain (x, y), where β does not vanishes. The covariant differentiation with respect to Levi-Civita connection $\gamma^i_{jk}(x)$ of R^n is denoted by (|). Let us denote
$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_j = b^r r^r_j, \; s_j = b_r s^r_j, \; b^i = a^{ir} b_r,$$
$$b^2 = a^{rs} b_r b_s.$$

Definition 2.2. A Finsler space is called Berwald space, if G^i_{jk} are the functions of position alone, i.e., Berwald connection $B\Gamma$ if linear.

The coefficients $G^i_{jk}(x, y)$ of Berwald connection can be computed from the sprays G^i as follows:
$G^i_j(x, y) = G^i_{(j)}$ and $G^i_{jk}(x, y) = G^i_{j(k)}$.

The Berwald connection $B\Gamma = \left(G^i_{jk}, G^i_j\right)$ of F^n plays a very important in the present article. According to [5], B^i_{jk} denote the difference tensor of G^i_{jk} from γ^i_{jk} of a Finsler space F^n are written in the form
$$G^i_{jk} = \gamma^i_{jk}(x) + B^i_{jk}(x, y). \qquad (2.2)$$
By transvecting the above by y^k and y^l, we have the functions $G^i(x, y)$ of F^n with (α, β)-metric are in the form
$$2G^i = \gamma^i_{00} + 2B^i$$
$$B^i = \frac{E}{\alpha} y^i + \alpha \frac{L_\beta}{L_\alpha} s^i_0 - \alpha \frac{L_{\alpha\alpha}}{L_\alpha} C^* \left[\frac{y^i}{\alpha} - \frac{\alpha}{\beta} b^i\right] \qquad (2.3)$$
where $E = \frac{\beta L_\beta}{L} C^*$,
$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}$ and
$\gamma^2 = b^2\alpha^2 - \beta^2$

Here, B^i is called the difference vector of a Finsler space with an (α, β)-metric. Here the subscript '0' means contraction by y^i. We use the following lemma later:

Lemma 2.1. [2] If $\alpha^2 \equiv 0(mod\,\beta)$, that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case, we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.

Throughout this paper, we consider that $\alpha^2 \neq 0(mod\beta)$.

3. Finsler space with (α, β)-metric of Berwlad type

The present section is devoted to find the condition for a Finsler space F^n with the metric (1.1) to be a Berwald space.

The partial derivatives of (1.1) with respect to α and β are as follows:

$$\left.\begin{aligned} L_\alpha &= \frac{2\mu\alpha^3-\nu\beta^3}{\alpha^2\beta}, L_\beta = \frac{2\nu\beta^3-\mu\alpha^3}{\alpha\beta^2},\\ L_{\alpha\alpha} &= \frac{2\mu\alpha^3+2\nu\beta^3}{\alpha^3\beta}, L_{\beta\beta}\ \frac{2\mu\alpha^3+2\nu\beta^3}{\alpha\beta^3},\\ L_{\alpha\beta} &= \frac{(-2\mu\alpha^3-\nu\beta^3)}{\alpha^2\beta^2}\end{aligned}\right\} \quad (3.1)$$

From (2.3), the Berwald connection $B\Gamma = \left(G^i_{jk}, G^i_j, 0\right)$ of F^n with (α, β)-metric is given by

$$G^i_j = \dot{\partial}_j G^i = \gamma^i_{0j} + B^i_j$$
$$G^i_{jk} = \dot{\partial}_k G^i_j = \gamma^i_{jk} + B^i_{jk}$$

where $B^i_j = \partial_j B^i$ and $B^i_{jk} = \partial_k B^i_j$.

Now the difference tensor B^i_{jk} are uniquely determined by

$$L_\alpha B^k_{ji} y^j y_k + \alpha\, L_\beta\left(B^k_{ji} b_k - b_{j|i}\right)y^j = 0, \quad (3.2)$$

where $B_{jki} = a_{kr}B^r_{ji}$.

Plugging (3.1) in (3.2), we have

$$(2\mu\alpha^3\beta - \nu\beta^4)B_{jki}y^j y^k + \alpha^2(2\nu\beta^3 - \mu\alpha^3) \cdot \left(B_{jki}b^k - b_{j|i}\right)y^j = 0, \quad (3.3)$$

where $B_{jki} = a_{kr}B^r_{ji}$.

We suppose that F^n is a Berwald space, then B^i_{jk} and $b_{i|j}$ are functions of position alone. Then (3.3) is separated as rational and irrational terms in $\left(y^i\right)$ as follows:

$$-\nu\beta^4 B_{jki}y^j y^k + 2\nu\alpha^2\beta^3\left(B_{jki}b^k - b_{j|i}\right)y^j + \alpha\{2\mu\alpha^2\beta\ B_{jki}y^j y^k - \mu\alpha^4\left(B_{jki}b^k - b_{j|i}\right)y^j\} = 0, \quad (3.4)$$

which yields two equations

$$-\nu\beta\ B_{jki}y^j y^k + 2\nu\alpha^2\left(B_{jki}b^k - b_{j|i}\right)y^j = 0, \quad (3.5)$$

$$2\mu\beta B_{jki}y^j y^k - \mu\alpha^2\left(B_{jki}b^k - b_{j|i}\right)y^j = 0. \quad (3.6)$$

Substituting (3.6) in (3.5), we have

$$3\nu\beta\ B_{jki}y^j y^k = 0, \quad (3.7)$$

which implies $B_{jki}y^j y^k = 0$. Thus, we have $B_{jki} + B_{kji} = 0$. Since B_{jki} is symmetric in (j, i), we get $B_{jki} = 0$ easily, and from (3.5) or (3.6), we have

$$b_{j|i} = 0. \quad (3.8)$$

Conversely, if $b_{j|i} = 0$, then $B_{jki} = 0$ are uniquely determined from (3.3).

Thus we state that:

Theorem 3.1. A Finsler space with a special (α, β)-metric (1.1) is Berwlad space if and only if $b_{j|i} = 0$.

4. Main scalar of two dimensional Finsler space with (α, β)-metric

The main scalar I of two dimensional Finsler space F^2 with the (α, β)-metric is given by

$$\epsilon I^2 = \left(\frac{L}{\alpha}\right)^4\left[\frac{\gamma^2(T_2)^2}{4T^3}\right] \quad (4.1)$$

where ϵ is signature of the space, $\gamma^2 = b^2\alpha^2 - \beta^2$,

$$T = P(P + P_0 b^2 + P_{-1}\beta) + \{P_0 P_{-2} - (P_{-1})^2\}\gamma^2$$

and $T_2 = \frac{\partial T}{\partial\beta}$ (4.2)

$$\left.\begin{aligned} &P = LL_1\alpha^{-1}, P_0 = LL_{22} + (L_2)^2,\\ &P_{-1} = (LL_{12} + L_1 L_2)\alpha^{-1},\\ &P_{-2} = L\alpha^{-2}(L_{11} - L_1\alpha^{-1}) + L_1^2\alpha^{-2}\end{aligned}\right\} \quad (4.3)$$

According to [3], If g denote the determinant of matrix (g_{ij}) and 'a' denote the matrix (a_{ij}) then in n-dimensional Finsler space with (α,β)-metric, we have

$$g=\left[p^{(n-2)}T\right]a. \tag{4.4}$$

Plugging (3.1) in (4.3), we have

$$\left.\begin{aligned} P&=\frac{2\mu^2\alpha^6+\mu\,\nu\alpha^3\beta^3-\nu^2\beta^6}{\alpha^4\beta^2},\\ P_0&=\frac{3\mu^2\alpha^6+6\nu^2\beta^6}{\alpha^2\beta^4},\\ P_{-1}&=\frac{-4\mu^2\alpha^6+\mu\nu\alpha^3\beta^3-4\nu^2\beta^6}{\alpha^4\beta^3},\\ P_{-2}&=\frac{4\mu^2\alpha^6-\mu\nu\alpha^3\beta^3+4\nu^2\beta^6}{\alpha^6\beta^2}\end{aligned}\right\} \tag{4.5}$$

For a two-dimensional Finsler space with (α,β)-metric, we have

$$\frac{g}{a}=T=\left(\frac{L}{\alpha}\right)^3\left(L_\alpha+\frac{L_{\beta\beta}}{\alpha}\gamma^2\right). \tag{4.6}$$

Plugging (4.5) in to (4.2) we have

$$T=\frac{T_3}{\alpha^8\beta^6} \text{ and } T_2=\frac{T_4}{\alpha^8\beta^7} \tag{4.7}$$

where,

$$\begin{aligned}T_3=&\,2b^2\mu^4\alpha^{14}+8b^2\mu^3\nu\alpha^{11}\beta^3-3\mu^3\nu\alpha^9\beta^5\\ &+12b^2\mu^2\nu^2\alpha^8\beta^6+8b^2\mu\nu^3\alpha^5\beta^9\\ &-9\mu^2\nu^2\alpha^6\beta^8-9\mu\nu^3\alpha^3\beta^{11}\\ &+2b^2\nu^4\alpha^2\beta^{12}-3\nu^4\beta^{14},\end{aligned} \tag{4.8}$$

$$\begin{aligned}T_4=&-12b^2\mu^4\alpha^{14}-24b^2\mu^3\nu\alpha^{11}\beta^3\\ &+3\mu^3\nu\alpha^9\beta^5+24b^2\mu\nu^3\alpha^5\beta^9\\ &-18\mu^2\nu^2\alpha^6\beta^8-45\mu\nu^3\alpha^3\beta^{11}\\ &+12b^2\nu^4\alpha^2\beta^{12}-24\nu^4\beta^{14}.\end{aligned} \tag{4.9}$$

Now plugging (4.7) in (4.1), the main scalar of two dimensional Finsler space becomes

$$\epsilon I^2=\frac{\left(\mu\alpha^3+\nu\beta^3\right)^4\gamma^2(T_4)^2}{4(T_3)^3}. \tag{4.10}$$

Thus, we state that:

Theorem 4.2. The main scalar of two dimensional Finsler space with (α,β)-metric (1.1) is given by (4.10).

5. Geodesic equation of a Finsler space with (α,β)-metric

For a given Finsler metric $F=F(x,y)$, the geodesics of F satisfy the following ODE's:

$$\frac{d^2x^i}{ds^2}+2G^i\left(x,\frac{dx}{ds}\right)=0, \tag{5.1}$$

where $G^i=G^i(x,y)$ is called the geodesic coefficient, which is given by

$$G^i=\frac{1}{4}g^{il}\left\{[F^2]_{x^m y^l}y^m-[F^2]_{x^l}\right\}.$$

With respect to a parameter 't', equation (5.1) can be written as

$$\frac{d^2x^i}{dt^2}+2G^i\left(x,\frac{dx}{dt}\right)=-\frac{t''}{t'}\frac{dx^i}{dt}, \tag{5.2}$$

where $t'=\frac{dt}{ds}$.

$$\frac{d^2x^i}{dt^2}+\gamma^i_{00}+\frac{2L_\beta}{L_\alpha}s^i_0+\frac{2(LL_{\alpha\alpha}E)}{L_\alpha L_\beta\,\beta^2}P^i=0, \tag{5.3}$$

where $P^i=b^i-\beta\alpha^2y^i$.

Now C^* and E becomes

$$\left.\begin{aligned}C^*&=\frac{\alpha\left[\beta\left(2\mu\alpha^3-\nu\beta^3\right)r_{00}-2\alpha^2\left(2\nu\beta^3-\mu\alpha^3\right)s_0\right]}{2[2b^2\mu\alpha^5+2b^2\nu\alpha^2\beta^3-3\nu\beta^5]}\\ E&=\frac{\left(\alpha\left(2\nu\beta^3-\mu\alpha^3\right)\begin{bmatrix}\beta\left(2\mu\alpha^3-\nu\beta^3\right)r_{00}\\-2\alpha^2\left(2\nu\beta^3-\mu\alpha^3\right)s_0\end{bmatrix}\right)}{2(\mu\alpha^3+\nu\beta^3)[2b^2\mu\alpha^5+2b^2\nu\alpha^2\beta^3-3\nu\beta^5]}\end{aligned}\right\} \tag{5.4}$$

Now plugging (3.1) and (5.4) in (5.3), we have the equation of geodesic for a Finsler space with the metric (1.1) is as follows

$$\begin{aligned}&\frac{d^2x^i}{dt^2}+\gamma^i_{00}+\frac{\left(2\alpha\left(2\nu\beta^3-\mu\alpha^3\right)\right)}{\beta\left(2\mu\alpha^3-\nu\beta^3\right)}\,s^i_0\\ &+\frac{\left(2\mu\alpha^3+2\nu\beta^3\right)\begin{bmatrix}\beta\left(2\mu\alpha^3-\nu\beta^3\right)r_{00}\\-2\alpha^2\left(2\nu\beta^3-\mu\alpha^3\right)s_0\end{bmatrix}}{\beta(2\mu\alpha^3-\nu\beta^3)[2b^2\mu\alpha^5+2b^2\nu\alpha^2\beta^3-3\nu\beta^5]}P^i=0,\end{aligned} \tag{5.5}$$

Thus, we state that

Theorem 5.3. With respect to an arc length 't' in the associated Riemannian space $R^n=(M^n,\alpha)$, the equation of geodesic of a Finsler space F^n with the metric (1.1) is in the form of (5.5), where $\gamma^i_{jk}(x)$ are Christoffel symbols of R^n.

6. Conclusion

In this article, we have considered a special (α,β)-metric which is a linear combination of two (α,β)-metrics L_1 and L_2 is in the form of

$$L=\mu\left(\frac{\alpha^2}{\beta}\right)+\nu\left(\frac{\beta^2}{\alpha}\right);\ \mu,\nu=constant. \quad (6.1)$$

where α is Riemannian metric and β is differential 1-form. With this metric, we have found Berwald connection and the condition under which a Finsler space with this metric is a Berwald space. Further, we investigated that the Main Scalars of two dimensional Finsler space with this metric. And finally, we are studied the geodesic equation of Finsler space with this metric. In this regard, we have obtained the following conclusions:

1. A Finsler space with a special (α,β) -metric (6.1) is Berwlad space if and only if $b_{j;i}=0$.
2. The main scalar of two dimensional Finsler space with (α,β)-metric (6.1) is given by (4.10).
3. The equation of geodesic of a Finsler space F^n with the metric (6.1) is in the form of (5.5), where $\gamma^i_{jk}(x)$ are Christoffel symbols of R^n.

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