A Study on Pseudo Quasi-Conformal Curvature Tensor in K-Contact Manifolds

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***Abstract*— The paper deals with the study on Pseudo quasi-conformal curvature tensor in K-contact manifolds and it is shown that the manifold is Einstein.**

Keywords: Conservative, Conformal curvature tensor, Einstein, K-contact manifold.

I. INTRODUCTION

Geometry of contact metric manifolds is important because of their applications, for instance in the theory of Einstein metrics K-contact manifolds have nice topological properties. Two important classes of contact manifolds are K-contact manifolds and Sasakian manifolds ([3], [4], [12]). K-contact and Sasakian manifolds have been studied by several authors such as ([1], [7], [9], [10]). It is known that if the characteristic vector field of a contact metric manifold is Killing vector field then the manifold is called a K-contact manifold. It is well known that every Sasakian manifold is K-contact, but the converse is not true, in general. However a three-dimensional K-contact manifold is Sasakian.

In [14], Yano and Sawaki defined and studied a tensor field W called quasi-conformal curvature tensor on a Riemannian manifold which includes both the conformal curvature tensor and concircular curvature tensor as special cases. In [13], A.A. Shaikh and Sanjib Kumar Jana introduced and studied the pseudo quasi-conformal curvature tensor and it is a generalization of conformal, quasi-conformal [1], concircular and projective curvature tensor as special cases. Recently, D.G. Prakasha et.al [8] and Satyabrota Kundu [11] studied different properties of pseudo quasi-conformal curvature tensor in P-Sasakian manifolds.

Motivated by the above work in this research article we studied pseudo quasi-conformal curvature tensor in K-contact manifolds:

The paper is organized as follows: Section 2 and 3 deals with preliminaries and the example of K-contact manifold. Section 4 deals with the pseudo quasi-conformally flat curvature tensor in K-contact manifold. In section 5 we study $\tilde{W}(\xi, X) \cdot S = 0$ in K-contact manifold. Section 6 is devoted to the study pseudo quasi-conformally conservative K-contact manifold.

II. PRELIMINARIES

An n-dimensional differentiable manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η on M satisfying

(2.1) $\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0,$
$\eta(\xi) = 1, \quad \eta \cdot \phi = 0.$

If g is a Riemannian metric with almost contact structure that is,

(2.2) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$
$\eta(X) = g(X, \xi).$

Then M is called an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) and denoted by (M, ϕ, ξ, η, g).

If on (M, ϕ, ξ, η, g) the exterior derivative of 1-form η satisfies,

$$d\eta(X, Y) = g(X, \phi Y).$$

Then (M, ϕ, ξ, η, g) is said to be a contact metric manifold.

If moreover ξ is Killing vector field, then M is called a K-contact manifold. A K-contact manifold is called Sasakian, if the relation

(2.3) $(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$

holds, where ∇ denotes the covariant differentiation with respect to g. From (2.3), we get

(2.4) $\nabla_X \xi = -\phi X, \ (\nabla_X \eta)Y = g(X, \phi Y).$

In a K-contact manifold M the following relations holds:

(2.5) $g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y),$
(2.6) $R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$
(2.7) $R(\xi, X)\xi = \eta(X)\xi - X,$
(2.8) $S(X, \xi) = (n-1)\eta(X),$

for any vector fields X, Y. Where R is the Riemannian curvature tensor and S is the Ricci tensor of the manifold M.

The pseudo quasi-conformal curvature tensor [13] $\tilde{W}$ of type $(1,3)$ on a K-contact manifold is defined by

$$(2.9)\ \tilde{W}(X,Y)Z = (p+d)R(X,Y)Z + \left[q - \frac{d}{n-1}\right][S(Y,Z)X - S(X,Z)Y] + q[g(Y,Z)QX - g(X,Z)QY] - \frac{r(p+2(n-1)q)}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],$$

for all vector fields X, Y, Z, where p, q, d are arbitrary constants not simultaneously zero, R is the Riemannian curvature tensor, S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature tensor of the manifold M.

In particular, if

- $p = q = 0, d = 1$ in (2.9) then $\tilde{W}$ reduces to the projective curvature tensor.
- $p \neq 0, q \neq 0, d = 0$ in (2.9) then $\tilde{W}$ reduces to the quasi-conformal curvature tensor.
- $p = 1,\ q = -\frac{1}{n-2},\ d = 0$ in (2.9) then $\tilde{W}$ reduces to the conformal curvature tensor.
- $p = 1,\ d = q = 0$ in (2.9) then $\tilde{W}$ reduces to the concircular curvature tensor.

III. EXAMPLE

Consider the 3-dimensional manifold $M = \{f(x,y,z) : (x,y,z) \in R^3\}$. Let (E_1, E_2, E_3) be linearly independent at each point of M

$$E_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\ E_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2) = g(E_2,E_3) = g(E_1,E_3) = 0,$$
$$g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1.$$

Where g is given by

$$g = [(1-y^2)dx \otimes dx + (1-x^2)dy \otimes dy + dz \otimes dz].$$

Let ξ be the vector field, η be the 1-form and ϕ be the $(1,1)$ tensor field defined by

$$\xi = \frac{\partial}{\partial z},\ \eta = dz + xdy - ydx,$$
$$\phi E_1 = -E_2,\quad \phi E_2 = E_1,\quad \phi E_3 = 0.$$

The linearity property of ϕ and g yields that

$$\eta(E_3) = 1,\quad \phi^2 U = -U + \eta(U)E_3,$$
$$g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W),$$

for all vector fields U, W on M. Thus for $E_3 = \xi$. The structure (ϕ, ξ, η, g) defines on M. By definition of Lie bracket, we have

$$[E_1, E_2] = -2E_3, [E_1, E_3] = [E_2, E_3] = 0.$$

Let ∇ be Levi-Civita connection with respect to the above metric g given by Koszul formula, that is

$$(3.1)\ 2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).$$

Then by Koszula formula, we have

$$(3.2)\ \nabla_{E_1}E_1 = 0, \nabla_{E_2}E_2 = 0, \nabla_{E_3}E_3 = 0,$$
$$\nabla_{E_1}E_2 = -E_3, \nabla_{E_2}E_1 = -E_3, \nabla_{E_2}E_3 = -E_1,$$
$$\nabla_{E_1}E_3 = E_2, \nabla_{E_3}E_1 = E_2, \nabla_{E_3}E_2 = -E_1.$$

Clearly one can see that (M, ϕ, ξ, η, g) is a K-contact manifold.

IV. PSEUDO QUASI-CONFORMALLY FLAT K-CONTACT MANIFOLD

In this section, we study the Pseudo quasi-conformally flat curvature tensor in K-contact manifold that is $\tilde{W} = 0$ then from (2.9), we get

$$(4.1)\ R(X,Y)Z = -\frac{1}{(p+d)}\left[q - \frac{d}{n-1}\right][S(Y,Z)X - S(X,Z)Y] - \frac{q}{(p+d)}[g(Y,Z)QX - g(X,Z)QY] + \frac{r(p+2(n-1)q)}{n(n-1)(p+d)}[g(Y,Z)X - g(X,Z)Y].$$

From (4.1), we have

$$(4.2)\ R(X,Y,Z,U) = -\frac{1}{(p+d)}\left[q - \frac{d}{n-1}\right][S(Y,Z)g(X,U) - S(X,Z)g(Y,U)] - \frac{q}{(p+d)}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U)] + \frac{r(p+2(n-1)q)}{n(n-1)(p+d)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

Putting $Y = Z = \xi$ in (4.2), we obtain

$$(4.3)\ R(X,\xi,\xi,U) = -\frac{n-1}{(p+d)}\left[q - \frac{d}{n-1}\right][g(X,U) - \eta(X)\eta(U)] - \frac{q}{(p+d)}[S(X,U) - (n-1)\eta(X)\eta(U)] + \frac{r(p+2(n-1)q)}{n(n-1)(p+d)}[g(X,U) - \eta(X)\eta(U)].$$

By using (2.6) in (4.3), we have

$$(4.4)\ S(X,U) = \left[\frac{r(p+2(n-1)q)}{n(n-1)q} - \frac{p}{q} - (n-1)\right]g(X,U) - \left[\frac{r(p+2(n-1)q)}{n(n-1)q} - \frac{p}{q} - 2(n-1)\right]\eta(X)\eta(U).$$

On contracting (4.4), we have

$$(4.5)\ r = n(n-1).$$

Substituting (4.5) in (4.4), we get

$$(4.6)\ S(X,U) = (n-1)g(X,U).$$

Hence, we state the following:

Theorem 4.1. An n-dimensional pseudo quasi-conformally at K-contact manifold is an Einstein manifold with constant scalar curvature tensor $n(n-1)$.

Next, by using (4.5) and (4.6) in (4.2), we obtain

$$(4.7)\ R(X,Y,Z,U) = [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

provided that $p + d \neq 0$. This implies that the K-contact manifold M is of constant curvature.

Conversely, if M is of constant curvature, then we get $\tilde{W}(X,Y)Z = 0$, that is M is pseudo quasi-conformally flat. Hence, we can state the following:

Theorem 4.2. An n-dimensional K-contact manifold is pseudo quasi-conformally flat if and only if it is the manifold of constant curvature provided that $p + 2(n-1)q \neq 0$ and $p + d \neq 0$.

V. K-CONTACT MANIFOLDS SATISFYING $\tilde{W}(\xi, X) \cdot S =$

In this section, we study $\tilde{W} \cdot S = 0$ in K-contact manifold:

The condition $\tilde{W}(\xi, X) \cdot S = 0$, implies that

$$(5.1)\; S(\tilde{W}(\xi,X)Y,Z) + S(Y,\tilde{W}(\xi,X)Z) = 0.$$

From (2.9), we get

$$(5.2)\; \tilde{W}(\xi,Y) = \left[(p+d) - \frac{r(p+2(n-1)q)}{n(n-1)}\right][g(Y,Z)\xi - \eta(Z)Y] + \left[q - \frac{d}{(n-1)}\right][S(Y,Z)\xi - (n-1)\eta(Z)Y] + q[(n-1)g(Y,Z)\xi - \eta(Z)QY].$$

By virtue of (5.2) in (5.1) and on simplification, we obtain

$$(5.3)\; \left[(p+d) - \frac{r(p+2(n-1)q)}{n(n-1)} + q(n-1)\right] [(n-1)g(X,Y)\eta(Z) - \eta(Y)S(X,Z) + (n-1)g(X,Z)\eta(Y) - \eta(Z)S(X,Y)] = 0.$$

Putting $Z = \xi$ in (5.3) and on simplification, we get

$$(5.4)\; S(X,Y) = (n-1)g(X,Y).$$

Provided $\left[(p+d) - \frac{r(p+2(n-1)q)}{n(n-1)} + q(n-1)\right] \neq 0$

On contracting (5.4), we have

$$(5.5)\; r = n(n-1).$$

Hence, we can state the following:

Theorem 5.3. A K-contact manifold satisfying $\tilde{W}(\xi, X) \cdot S = 0$ is an Einstein manifold and scalar curvature $r = n(n-1)$.

VI. PSEUDO QUASI-CONFORMALLY CONSERVATIVE K-CONTACT MANIFOLD

In this section, we study the conservative Pseudo Quasi-Conformal curvature tensor in K-contact manifold. Then we have

$$(6.1)\; div\, \tilde{W}(X,Y)Z = 0.$$

Differentiating (2.9) covariantly along the vector field U, we get

$$(6.2)\; (\nabla_U \tilde{W})(X,Y)Z = (p+d)(\nabla_U R)(X,Y)Z + \left[q - \frac{d}{(n-1)}\right][(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y] + q[g(Y,Z)(\nabla_U Q)X - g(X,Z)(\nabla_U Q)Y] - \frac{dr(U)}{n(n-1)}[p + 2(n-1)q][g(Y,Z)X - g(X,Z)Y].$$

On contracting (6.2), we obtain

$$(6.3)\; div\, \tilde{W}(X,Y)Z = (p+d)\, div\, R(X,Y)Z + \left[q - \frac{d}{(n-1)}\right][(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] - \left[\frac{p}{n(n-1)} - \frac{(n-4)q}{2n}\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)].$$

By (6.1) in (6.3), we have

$$(6.4)\; \left[p + q + \frac{d(n-2)}{(n-1)}\right][(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] - \left[\frac{p}{n(n-1)} - \frac{(n-4)q}{2n}\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)] = 0.$$

Putting $X = \xi$ in (6.4), we get

$$(6.5)\; \left[p + q + \frac{d(n-2)}{(n-1)}\right][(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)] - \left[\frac{p}{n(n-1)} - \frac{(n-4)q}{2n}\right][g(Y,Z)dr(\xi) - g(\xi,Z)dr(Y)] = 0.$$

We know that

$$(6.6)\; (\nabla_Y S)(\xi,Z) = YS(\xi,Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z) = S(\phi Y, Z) - (n-1)g(\phi Y, Z).$$

And

$$(6.7)\; (\nabla_Y S)(\xi,Z) = 0.$$

By using (6.6), (6.7) and $dr(\xi) = 0$ in (6.5), we get

$$(6.8)\; \left[p + q - \frac{d(n-2)}{(n-1)}\right][S(\phi Y,Z) - (n-1)g(\phi Y,Z)] = \left[\frac{p}{n(n-1)} - \frac{(n-4)q}{2n}\right]\eta(Z)dr(Y).$$

Replacing Z by ϕZ in (6.8), we obtain

$$(6.9)\; S(\phi Y, \phi Z) = (n-1)g(\phi Y, \phi Z),$$

provided $\left[p + q + \frac{d(n-2)}{(n-1)}\right] \neq 0$. On simplifying (6.9), we get

$$(6.10)\; S(Y,Z) = (n-1)g(Y,Z).$$

On contracting (6.10), we have

$$(6.11)\; r = n(n-1).$$

Hence, we state the following:

Theorem 6.4. If in a K-contact manifold the Pseudo quasi-conformal curvature tensor is conservative, then the manifold is Einstein and scalar curvature $r = n(n-1)$.

ACKNOWLEDGMENT

The authors are very grateful to the referees for their valuable suggestions and opinions.

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