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# Conditional Sandwiched Rényi Relative Entropy and its application in separability of single parameter families of 3, 4 qubit W and GHZ density functions

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***Abstract***

*An increase in activity has been seen recently in the study of entanglement in bipartite quantum systems. The entropic characterization of separability of composite quantum systems has attracted a lot of interest among diverse techniques. In this research article, a new form of conditional Rényi relative entropy (known as Conditional Sandwiched Rényi Relative Entropy (CSRRE) form) is used to determine the bipartite separability limit in single parameter families of mixed W- and GHZ- states. For N=3, 4 qubits, the non-entanglement limit in the 1: N-1 partition of these states determined using CSRRE criterion is more restrictive than the separability range discovered by Abe-Rajagopal's form of Tsallis entropy. It's established that1: N-1separabilty range determined using CSRRE approach precisely with the range determined by positivity of partial transpose criterion.*

***Keywords:*** *Entanglement, Rényi relative entropy, Mixed W- and GHZ- states, Positivity of partial transpose*

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## 1. Introduction

A lot of attention has lately been paid to characterizing the separability of composite quantum systems using the quantum entropic approach [1–9]. In a pure bipartite state, the von Neumann conditional entropy is negative, indicating that there is entanglement between the two parties. This shows that while it is untrue for separable states, in entangled states the global disorder is lesser than the local disorder [10]. Extended quantum conditional entropies are preferable to conditional von Neumann entropies for the investigation of global vs. local disorder in mixed states.

In order to examine bipartite separability in various one parameter families of mixed quantum states, a novel conditional version of generalized Sandwiched Tsallis Relative Entropy (CSTRE) has recently been used [11–14]. The bipartite non-entanglement limits that got through this novel Tsallis entropic criterion were more stringent than the non-entanglement limits that got through conventional entropic criteria. They really fitted with the the non-entanglement limits got according to Peres' Positivity of Partial Transpose (PPT) criteria. The search for new entropic criteria is encouraged by conditional Tsallis entropy's success in identifying the tougher non-entanglement limits.

In this study, the bipartite separability of N-qubit one parameter families of mixed W- and GHZ- states is determined using the conditional form of generalized Rényi relative entropy [15]. We also compare the findings to those obtained using the PPT criteria and the Abe-Rajagopal version of Tsallis entropy.

## 2. Rényi relative entropy and its new conditional form

The Rényi and Tsallis entropies, denoted, respectively, by $S^R(\rho)$ and $S^T(\rho)$, are as follows

$$S^R(\rho) = \frac{1}{1-q} \ln \mathrm{Tr}\,(\rho^q), \qquad (1)$$

$$S^T(\rho) = \frac{Tr\,(\rho^q)-1}{(1-q)}$$

Here $q$ is a non-zero, non-negative real parameter. In the limit $q \to 1$ both Rényi and Tsallis entropies reduce to von Neumann entropy.

For the density operator's $\rho$ and $\sigma$ the conventional quantum relative Rényi entropy is defined, as

$$D^R(\rho||\,\sigma) = \frac{\ln\, Tr(\rho^q\, \sigma^{1-q})}{q-1} \text{ for } q \in (0,1) \cup (1, \infty)$$

$$= \mathrm{Tr}\,[\,\rho(\,\ln\rho - \ln\sigma\,)] \quad for\ q \to 1. \qquad (2)$$

Independently, Wilde et al. [11] and Müller-Lennert et al. [12] presented a generalized version of quantum relative Rényi entropy:

$$\widetilde{D}^R(\rho||\,\sigma) = \frac{1}{q-1} \ln\,[\,Tr(\,\sigma^{1-\frac{q}{2q}}\rho\,\sigma^{1-\frac{q}{2q}})^q] \qquad (3)$$

for $q \in (0,1) \cup (1, \infty)$

The quantum relative Rényi entropy (3) reduces to the conventional value given by (2) whenever the density operator's $\rho$ and $\sigma$ commute; hence the current form is an extension to the non-commutative situation.

The conditional version of $\widetilde{D}^R(\rho_{AB}||\,\sigma)$ (we call it as Conditional Sandwiched Rényi Relative Entropy (CSRRE)) is now defined by taking $\sigma = I_A \otimes \rho_B$ (or $\rho_{AB} \otimes I_B$) where $\rho_B = \mathrm{Tr}_A(\rho_{AB})$ [ similarly $\rho_B = \mathrm{Tr}_B\,(\rho_{AB})$] the single party density matrix of the state $\rho_{AB}$. It is given by

$$\widetilde{D}^R(\rho_{AB}||\,\rho_B) = \frac{1}{(q-1)}\, Q(\rho_{AB}||\,\rho_B) \qquad (4)$$

Where $Q(\rho_{AB}||\,\rho_B) = \ln\,[\,Tr(\,\sigma^{\frac{1-q}{2q}}\,\rho\,\sigma^{\frac{1-q}{2q}})^q]$

One must build the unitary operator that diagonalizes the subsystem density matrix $\rho_B$ in order to evaluate the formula $Q(\rho_{AB}||\,\rho_B)$.

If the unitary operator $U_B$ that diagonalizes $\rho_B$, we have

$$\sigma_D = U\ \sigma^{\frac{1-q}{2q}}\ U^\dagger$$

$$\text{for } \sigma = I \otimes \rho_B\ ,\ \mathrm{U} = I \otimes U_B\ ,$$

$$\sigma_D = diag\,(\lambda_1^{\frac{1-q}{2q}}, \ldots \lambda_n^{\frac{1-q}{2q}})$$

Thus $Q(\rho_{AB}||\,\rho_B)$ in equation (4) simlifies to

$$Q(\rho_{AB}||\,\rho_B) = \mathrm{Tr}[(\sigma_D\, U\rho\, U^\dagger\, \sigma_D\,)^q] \qquad (5)$$

By evaluating the eigenvalues of $\sigma_D\, U\,\rho\, U^\dagger\, \sigma_D$ and adding them gives the quantity $Q(\rho_{AB}||\,\rho_B)$. Thus the sandwiched form of conditional Rényi relative $\widetilde{D}^R(\rho_{AB}||\,\rho_B)$ is obtained. In the section that follows, this form is now used to determine the separability range in the one-parameter families of the W and GHZ states.

## 3. One-parameter families of W and GHZ states and their separability.

Incorporating either a W state or a GHZ state, the symmetric one-parameter families of N qubit mixed states are provided by

$$\rho_N^{(W)}(x) = \left(\frac{1-x}{N+1}\right) P_N + x\,|W\rangle_N\langle W|$$

and

$$\rho_N^{(GHZ)}(x) = \left(\frac{1-x}{N+1}\right) P_N + x\,|GHZ\rangle_N\langle GHZ|$$

Here $0 \le x \le 1$, $P_N = \sum_M |\frac{N}{2}, M\rangle\,\langle\frac{N}{2}, M|$ are the symmetric subspace projectors of $N$-qubits formed by the $N + 1$ angular momentum states$|\frac{N}{2}, M\rangle$, M $= \frac{N}{2},\ \frac{N}{2} - 1, \ldots, -\frac{N}{2}$, pertaining to the highest value of total angular momentum $J = N/2$.

The AR q-conditional entropy has been used to determine the non-entanglement limits of the noisy one-parameter family of W and GHZ states, as shown in Ref. [8]. Their results for the two qubit states of $\rho_N^{(W)}(x)$ matches with the positive partial transpose (PPT) criterion [18] but for both W and GHZ state families the non-entanglement limits obtained by them is weaker than the limits obtained through PPT when $N \geq 3$.

It observed that the noncommutativity between density operator $\rho_{AB}$ with $\rho_B$ is crucial and although it is weaker than the PPT criterion, the range of separability provided through nonnegative values of the conditional sandwiched Rényi relative entropy is stricter than that derived through the AR q-conditional entropy.

For $\rho_3^{(W)}(x) \equiv \rho_{ABC}$ the matrix form of $\rho_{BC}$ obtained and also we arrive at $\sigma_D\, U\, \rho_3^{(W)}\, U^\dagger\, \sigma_D$ through the unitary matrix that diagonalizes it. Here we have

$$\sigma_D = I_2 \otimes \mathrm{diag}\left(\left(\frac{1}{3}\right)^{\frac{1-q}{2q}}, 0, \left(\frac{1-x}{3}\right)^{\frac{1-q}{2q}}, \left(\frac{1+x}{3}\right)^{\frac{1-q}{2q}}\right)$$

$$U = I_2 \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \ \rho = \rho_3^{(W)}(x)$$

The matrix $\sigma_D\, U\, \rho\, U^\dagger\, \sigma_D$ contains the non-zero eigenvalues as shown

$$\gamma_1 = \frac{3(1-x)3^{-1/q}}{4}, \ \gamma_2 = \frac{3(1-x)^{1/q}3^{-1/q}}{4},$$

$$\gamma_3 = \frac{3^{-1/q}\,(1+3x)[1+x+2(1-x)^{1/q}]}{4\,(1+x)},$$

$$\gamma_3 = \frac{3^{-1/q}\,[(1+x)(1-x)^{\frac{1}{q}}+2(1-x)(1+x)^{1/q}]}{4\,(1+x)}$$

The Conditional Sandwiched Rényi Relative Entropy in Eq. (4) may now be evaluated for various values of q, and one can derive $\widetilde{D}^R(\rho_{AB}||\,\rho_B)$ as a function of x.

The plots in Figs. 1 and 2 show the tighter non-entanglement limit for $\rho_3^{(W)}(x)$ in its A:BC partition for increasing values of q.

In the A:BC partition of the one-parameter family of three-qubit W states, the non-entanglement limit 0 x 0.1547 obtained through the Conditional sandwiched Rényi relative entropy approach is in perfect agreement with that obtained from the partial transpose criterion, as can be seen from Figs. 1 and 2. It should be observed that for the A:BC partition of $\rho_3^{(W)}(x)$, AR q-conditional entropy gives a weaker non-entanglement limit $0 \leq x \leq 0.2$ .

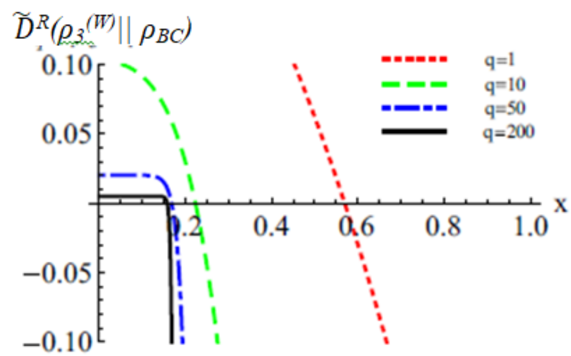


**Figure 1:** (Color online) CSRRE $\widetilde{D}^R(\rho_3^{(W)}||\,\rho_{BC})$ as a function of $x$ for the state $\rho_3^{(W)}$ for different q.

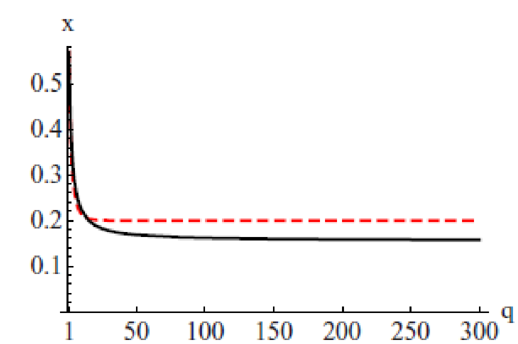


**Figure 2:** (Color online) The solid line indicates the implicit plot of $\widetilde{D}^R(\rho_3^{(W)}||\,\rho_{BC}) = 0$ as a function of $q$. The dashed line shows the implicit plot $S^T_q\,(A|BC) = 0$.

The non-entanglement limits in the A:BCD partition of the state is determined by evaluating the CSRRE for $\rho_4^{(W)}(x)$ in a manner similar to that described before.

The non-entanglement limits determined by the PPT criteria is fully consistent with the observation that $\rho_4^{(W)}(x)$ is separable for $x \leq 0.1124$. Using an implicit plot of $\widetilde{D}^R(\rho_4^{(W)}||\,\rho_{BCD}) = 0$ to represent our finding for $\rho_4^{(W)}(x)$, we compare it to the AR q-conditional entropy

$S^T_q$ $(A|BCD)$ = 0in Fig. 3. It is important to note that the noncommuting nature of the state $\rho_N^{(W)}(x)$ and its reduced equivalents $I_A \otimes \rho_{N-1}$ make them excellent test cases for the current CSRRE separability criterion.

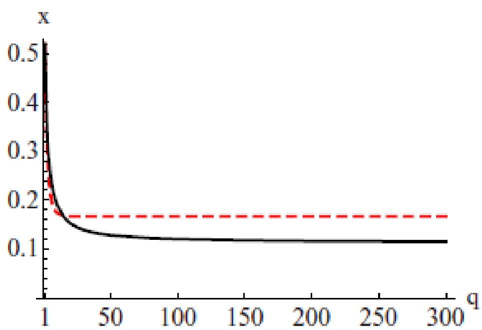


**Figure 3:** Solid line indicates the implicit plot of $\widetilde{D}^R(\rho_4^{(W)}||\ \rho_{BCD})$ = 0 and dashed line is for $S^T_q$ $(A|BCD)$ = 0 for different values of $q$..

In Ref [8], the non-entanglement limit for N qubit GHZ states is calculated using AR q-conditional entropy, and only the A:BC partition's non-entanglement limit was found to fulfill the PPT criterion's $\rho_3^{(GHZ)}(x)$ value. It should be noted that the PPT criterion and the AR q-conditional entropy criterion both indicate that the non-entanglement limit in the A: BC partition of $\rho_3^{(GHZ)}(x)$ is [0, 0.1428]. The same non-entanglement limit is seen in an explicit assessment using the CSRRE technique, demonstrating that the PPT range defines the boundary of the CSRRE separability domain.

Only in the A:BCD partition of $\rho_4^{(GHZ)}(x)$do the non-entanglement limits derived using the PPT criteria and the AR q-conditional entropy technique coincide for N = 4 as well. Here, we list the non-entanglement limit achieved by the current CSRRE technique and demonstrate that they are identical to those obtained by the AR q-conditional approach in all conceivable partitions of the state $\rho_4^{(GHZ)}(x)$. Our findings for the various partitions of the W and GHZ one-parameter families are summarized in Table I. In some of the noncommuting cases, such as in $\rho_{N=3,4}^{(W)}(x)$ in one of their A:BC and A:BCD partitions, the CSRRE approach matches the PPT criterion and produces a non-entanglement limit that is either equal to or more stringent than the range obtained through the AR q-conditional entropy.

**TABLE I.** List of non-entanglement limit of various states through PPT and different entropic criteria.

| **Quantum state** | **von Neumann conditional entropy** | **AR *q*-conditional entropy** | **CSRRE** | **PPT** |
|---|---|---|---|---|
| | | **$\rho_3^{(W)}(x)$** | | |
| *A:BC* partition | {0,0.5695} | {0,0.2} | {0,0.1547} | {0,0.1547} |
| *AB:C* partition | {0,0.7645} | {0,0.4286} | {0,0.3509} | {0,0.1547} |
| | | **$\rho_3^{(GHZ)}(x)$** | | |
| *A:BC* partition | {0,0.5482} | {0,0.1428} | {0,0.1428} | {0,0.1428} |
| *AB:C* partition | {0,0.7476} | {0,0.3333} | {0,0.3333} | {0,0.1428} $\rho_4^{(W)}(x)$ |
| | | **$\rho_4^{(W)}(x)$** | | |
| *A:BCD* partition | {0,0.5193} | {0,0.1666} | {0,0.1123} | {0,0.1123} |
| *AB:CD* partition | {0,0.6560} | {0,0.2105} | {0,0.2105} | {0,0.0808} |
| *ABC:D* partition | {0,0.8222} | {0,0.5454} | {0,0.4174} | {0,0.1123} |
| | | **$\rho_4^{(GHZ)}(x)$** | | |
| *A:BCD* partition | {0,0.4676} | {0,0.0909} | {0,0.0909} | {0,0.0909} |
| *AB:CD* partition | {0,0.6560} | {0,0.2105} | {0,0.2105} | {0,0.0625} |
| *ABC:D* partition | {0,0.7868} | {0,0.375} | {0,0.375} | {0,0.0909} |

## 4. Conclusion

The eigenvalues of the composite state and its subsystems are necessary for determining entanglement using Rényi and Tsallis conditional entropies [1–9]. Global disorder is less for the separable states than local disorder [10], as shown by the positivity of their conditional entropies. Clearly, entanglement is shown by the negative conditional entropies. However, the eigenvalue-based approach is just sufficient but not necessary to find entanglement. The separability of noisy one-parameter families of three- and four-qubit W and GHZ states has been investigated using the CSRRE. The outcomes were compared with those achieved using Peres' PPT criteria and AR q-conditional entropy. The CSRRE was demonstrated to be superior than AR q-conditional entropy; nevertheless, the separability range is constrained by the PPT criteria. The outcomes of our study are presented in Table I. The findings unambiguously show that the CSRRE technique for the 3 and 4 qubit one parameter family of W and GHZ states is either the same as or weaker than the PPT criteria. It is currently unclear how to determine the separability range for more qubits.

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