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Nonholonomic Frames for Finsler space with Special (α, β)-metrics

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Abstract

In the present paper, we determine the nonholonomic Frames for Finsler space with special (α, β)-metrics of type $L(\alpha, \beta) = (\alpha + \beta)\left(\frac{\alpha^2}{\alpha-\beta}\right)$ and $L(\alpha, \beta) = (\alpha + \beta)\left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)$ and also we observed the nonholonomic frames expesses as a Guage Transformation of Finsler metric.

Keywords: *(α, β)-metrics, Nonholonomic frames, Guage transformations, Beil metric, Finsler space.*

1. Introduction

The concept of theory of gauge transformation has been established in the context of Finsler space by G. S. Asanov and his co-researchers (1985-1989) [1], here interesting thing is that the theory of guage transformation the Finsler tangent vectors are considered as independent variables are attached to points in space-time. The homogeneous transformations of the tangent space are called guage transformations. In 1982, P. R. Holland worked on a unified (formalism) field theory that uses a nonholonomic Finsler frame on space-time is a sort of plastic deformation by considering the motion of charged particles in an electromagnetic field[2,3,4]. Nonholonomic frames have studied by so many physicists and geometricians about the motion of charged particles in electromagnetic field theory. In 1951, Y. Katsurada [7] introduced the theory of nonholonomic system in Finsler geometry. In 1995, R. G. Beil [5,6] have worked on a guage transformation considered as a nonholonomic frame on the tangent bundle of a four-dimensional base manifold. This introduces that there is unified approach to gravitation and guage symmetries. I. Bucataru[2], in his research he discussed about the how Beil metric is used in deformation of Riemannian metric and also in nonholonomic frame. For finding the nonholonomic frame he consider the most general case of Beil's metric. In this case, the Generalized Lagrange metric (in short, GL-metric) is known as Beil metric. In this article, evaluated the nonholonomic finslerian deformation with the some distinct special (α, β)-metrics are as follows:

1. $L(\alpha, \beta) = (\alpha + \beta)\left(\frac{\alpha^2}{\alpha-\beta}\right)$, i.e., product of Randers metric and Matsumoto metric.
2. $L(\alpha, \beta) = (\alpha + \beta)\left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)$, i.e., product of Randers metric and first approximate Matsumoto metric.

2. Preliminaries

Definition 2.1. Let U be an open set of TM and $V_i : u \in U \to V_i(u) \in V_uTM, i \in \{1, 2, ..., n\}$ be a vertical frame over U. If $V_i(u) = V_j^i(u)\frac{\partial}{\partial y^j}|_u$, then $V_i^j(u)$ are the entries of invertible matrix for all $u \in U$. Denote by $V_k^j(u)$ the inverse of this matrix. This means that: $V_j^i V_k^j = \delta_k^j$. We call V_j^i a nonholonomic Finsler Frame.

Definition 2.2. A Finsler space $F^n = (M, F(x,y))$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F: TM \to R$ is given by

$$F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\} \quad (2.1)$$

where $\alpha^2(x,y) = a_{ij}(x)y^i y^j$, α is a Riemannian metric on the manifold M and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

Definition 2.3. [2] A generalized Lagrange metric is a metric g on the vertical subbundle VTM of the tangent space TM. This means that for every $u \in TM$, $g_u: V_u\, TM \times V_u\, TM \to R$ is bilinear, symmetric, of rank n and of constant signature. A pair $GL^n = (M, g)$, with g a GL-metric is called a Generalized Lagrange space. If $(\pi^{-1}(U), \phi = (x^i, y^i)$ is an induced local chart at $u = (x,y) \in TM$, we denote by $g_{ij}(u) = g_u\left(\frac{\partial}{\partial y^i}|_u, \frac{\partial}{\partial y^j}|_u\right)$. Then a GL-metric may be given by a collection of functions $g_{ij}(x,y)$ such that we have: $rank(g_{ij}) = n, g_{ij}(x,y) = g_{ji}(x,y)$; the quadratic form $g_{ij}(x,y)\xi^i \xi^j$ has constant signature on TM; if another local chart $(\pi^{-1}(V), \phi = (\widetilde{x^i}, \widetilde{y^i})$ at $u \in TM$ is given and $\widetilde{g_{kl}}(x,y) = g_u\left(\frac{\partial}{\partial \widetilde{y^k}}|_u, \frac{\partial}{\partial \widetilde{y^l}}|_u\right)$ then g_{ij} and are related by

$$g_{ij} = \frac{\partial \widetilde{x^k}}{\partial y^i}\frac{\partial \widetilde{x^l}}{\partial y^j}\, \widetilde{g_{kl}}.$$

Further consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space (M, F). Taking into account the homogeneity of α and F we have the following formulae:

$$p^i = \frac{1}{\alpha}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j};$$

$$P_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i};$$

$$l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^j};$$

$$l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = p_i + b_i\, l^i = \frac{1}{L}p^i;$$

$$l^i l_i = p^i p_i = 1;\; l^i p_i = \frac{\alpha}{L};$$

$$p^i l_i = \frac{L}{\alpha};$$

$$b_i P^i = \frac{\beta}{\alpha}; b_i l^i = \frac{\beta}{L} \quad (2.2)$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_i p_j + p_i b_j + b_i b_j - \frac{\beta}{\alpha}p_i p_j = \frac{L}{\alpha}(a_{ij} - p_i p_j) + l_i l_j. \quad (2.3)$$

Theorem 2. 1. [12] For a Finsler space (M, F) consider the metric with the entries:

$$Y_j^i = \sqrt{\frac{\alpha}{L}}\left(\delta_j^i - l^i l_j + \sqrt{\frac{\alpha}{L}}p^i p_j\right); \quad (2.4)$$

defined on TM. Then $Y_j = Y_j^i\left(\frac{\partial}{\partial y^i}\right), j \in 1, 2, 3, ..., n$ is a nonholonomic frame.

Theorem 2. 2. [9] With respect to frame the holonomic components of the Finsler metric tensor $a_{\alpha\beta}$ is the Randers metric g_{ij}, i.e.,

$$g_{ij} = Y_i^\alpha Y_j^\beta a_{\alpha\,\beta}. \quad (2.5)$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j$, $\beta^i = a^{ij}b_j$ and so on. For a Finsler space with (α,β)-metric $F^2(x,y) = L\{\alpha(x,y),\beta(x,y)\}$ we have the Finsler invariants [13].

$$\rho = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha},$$

$$\rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2},$$

$$\rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha\, \partial \beta},$$

$$\rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right),$$

(2.6)

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with (α,β)-metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0.$$

(2.7)

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)-metric is given by[13]

$$g_{ij}(x,y) = \rho\, a_{ij}(x) + \rho_0\, b_i(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_i y_j. \quad (2.8)$$

From (2.8) we can see that g_{ij} is the result of two Finsler deformations:

1) $a_{ij} \to h_{ij} = \rho\, a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i) \cdot (\rho_{-1}b_j + \rho_{-2}y_j)$
2) $h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho_{-1}^2)b_i b_j$

The nonholonomic Finsler frame that corresponding to the theorem (7.9.1) in [12], given by,

$$X_j^i = \sqrt{\rho}\,\delta_j^i - \frac{1}{B^2}\left\{\sqrt{\rho} + \sqrt{\rho + \frac{B^2}{\rho_{-2}}}\right\}(\rho_{-1}\, b^i + \rho_{-2}\, y^i)(\rho_{-1}b_j + \rho_{-2}\, y_j), \quad (2.9)$$

where

$$B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}\, b_j + \rho_{-2}\, y_j) = \rho_{-1}^2 b^2 + \beta\, \rho_{-1}\rho_{-2}.$$

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^k X_j^l a_{kl}.$$

(2.10)

Again, the frame that corresponds to the II^{nd} deformation (2.9) given by,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left\{1 \pm \sqrt{1 + \frac{(\rho_{-2}\, C^2)}{(\rho_0\rho_{-2} - \rho_{-1}^2)}}\right\} b^i b_j,$$

(2.11)

where

$$C^2 = h_{ij}b^i b^j = \rho\, b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2.$$

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn} = Y_m^i Y_n^j h_{ij}.$$

(2.12)

Theorem 2.3. [12] Let $F^2(x,y) = L(\alpha(x,y),\beta(x,y))$ be the metric function of a Finsler space with (α,β)-metric for which the condition (2.7) is true. Then

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) and (2.12) respectively.

3. Nonholonomic Frames for Finsler Space with (α,β)-metric

Here we have two kind of metrics combination, such as product of Randers and Matsumoto metric, another one is product of Randers and first approximate of Matsumoto metric.

3.1. Nonholonomic frame for $L(\alpha,\beta) = (\alpha + \beta)\left(\frac{\alpha^2}{\alpha-\beta}\right)$

In the first case, for a Finsler space with the fundamental function $L = (\alpha + \beta)\left(\frac{\alpha^2}{\alpha-\beta}\right) = \frac{\alpha^2(\alpha+\beta)}{\alpha-\beta}$ the Finsler invariants (2.6) are given by

$$\rho = \frac{\alpha^2-\alpha\,\beta-\beta^2}{(\alpha-\beta)^2},$$

$$\rho_0 = \frac{2\alpha^3}{(\alpha-\beta)^2},$$

$$\rho_{-1} = \frac{\alpha(\alpha-3\beta)}{(\alpha-\beta)^3},$$

$$\rho_{-2} = \frac{\beta\,(3\beta-\alpha)}{\alpha(\alpha-\beta)^3},$$

$$B^2 = \frac{(\alpha-3\beta)^2(\alpha^2b^2-\beta^2)}{(\alpha-\beta)^6}. \qquad (3.1)$$

$$X_j^i = \sqrt{\frac{\alpha^2-\alpha\,\beta-\beta^2}{(\alpha-\beta)^2}}\,\delta_j^i - \frac{\alpha^2}{\alpha^2b^2-\beta^2}$$

$\left(\sqrt{\frac{\alpha^2-\alpha\beta-\beta^2}{(\alpha-\beta)^2}} \pm\right.$
α2−αβ−β2α β
−β2+α3β−αα2b2−β2βα−β3

$$\left(b^i - \frac{\beta}{\alpha^2}y^i\right)\left(b_j - \frac{\beta}{\alpha^2}y_j\right),$$
(3.2)

Again using (3.1) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1+\frac{\beta(\alpha-\beta)^2C^2}{\alpha^3}}\right)b^ib_j,$$
(3.3)

where

$$C^2 = \left(\frac{\alpha^2-\alpha\beta-\beta^2}{(\alpha-\beta)^2}\right)b^2 - \frac{(\alpha-3\beta)(\alpha^2b^2-\beta^2)^2}{\alpha\,\beta(\alpha-\beta)^3}.$$

Theorem 3.4. Let $L = (\alpha + \beta)\left(\frac{\alpha^2}{\alpha-\beta}\right) = \frac{\alpha^2(\alpha+\beta)}{\alpha-\beta}$ be the metric function of a Finsler space with (α,β)-metric for which the condition (2.7) is true. Then

$$V_j^i = X_k^iY_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (3.2) and (3.3) respectively.

3.2. Nonholonomic frame for $L(\alpha,\beta) = (\alpha+\beta)\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right) = \left(\alpha^2+2\alpha\beta+2\beta^2+\right.$ β2α.

In the second case, for a Finsler space with the fundamental function $L = (\alpha+\beta)\left(\alpha+\right.$ β+β2α=α2+2αβ+2β2+β2α be the product of Randers metric and first approximate Matsumoto metric the Finsler invariants (2.6) are given by

$$\rho = 1+\frac{\beta}{\alpha}-\frac{\beta^3}{2\alpha^3},$$

$$\rho_0 = 2+\frac{3\beta}{\alpha},$$

$$\rho_{-1} = \frac{1}{\alpha}-\frac{3\beta^2}{2\alpha^3},$$

$$\rho_{-2} = \frac{(3\beta^3-2\alpha^2\beta)}{2\alpha^5},$$

$$B^2 = \frac{(2\alpha^2-3\beta^2)^2(\alpha^2\,b^2-\beta^2)}{4\alpha^8}. \qquad (3.4)$$

Using (3.4) in (2.10) we have,

$$X_j^i = \sqrt{\left(1+\frac{\beta}{\alpha}-\frac{\beta^3}{2\alpha^3}\right)}\,\delta_j^i - \frac{\alpha^2}{\alpha^2b^2-\beta^2}\left(\sqrt{1+\frac{\beta}{\alpha}-\frac{\beta^3}{2\alpha^3}} \pm\right.$$
3β2−2α2α2b2 −2β2+2βα3+β32α3β

$$\cdot\left(b^i-\frac{\beta}{\alpha^2}y^i\right)\left(b_j-\frac{\beta}{\alpha^2}y_j\right) \qquad (3.5)$$

again using (3.1) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1+\frac{2\alpha\beta C^2}{2\alpha^2+4\alpha\beta+3\beta^2}}\right)b^ib_j, \qquad (3.6)$$

where

$$C^2 = \left(\frac{2\alpha^3+2\alpha^2\beta-\beta^3}{2\alpha^3}\right)b^2 + \frac{3\beta^2-2\alpha^2}{2\alpha^5}\cdot(\alpha^2b^2-\beta^2)^2.$$

Theorem 3.5. Let $L = (\alpha + \beta)\Big(\alpha + \beta +$ $\beta 2\alpha = \alpha 2 + 2\alpha\beta + 2\beta 2 + \beta 2\alpha$ be the metric function of a Finsler space with (α, β)-metric for which the condition (2.7) is true. Then

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (3.5) and (3.6) respectively.

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