Abstract: The object of the present paper is to study irrotational conharmonic, concircular, $\mathcal{M}$-projective and quasi-conformal curvature tensors on $(\mathcal{LCS})_n$-manifold.

Key Words: Lorentzian manifold, irrotational, conharmonic curvature tensor, concircular curvature tensor, $\mathcal{M}$-projective curvature tensor, quasi-conformal curvature tensor.

AMS Subject Classification: 53C05, 53C20, 53C25, 53C50, 53D10.

Introduction: In 1989, Matsumoto [5] introduced a manifold $M$ with a Lorentzian almost paracontact structure $(\phi, \xi, \eta, g)$. Mihai and Rosca [6] defined the same concept independently and obtained several results on this manifold. The author [9] introduced Lorentzian almost paracontact manifold with a structure of the concircular type and such a manifold is said to be a $(\mathcal{LCS})_n$-manifold, which generalizes the notion of LP-Sasakian manifolds. The $(\mathcal{LCS})_n$-manifolds were studied with various curvature conditions by Venkatesha [14], Prakash [8], Yadav [15], Shaikh et al. ([10,11,12,13]) and others.

Let $M^n$ be a Lorentzian manifold admitting concircular vector field $\xi$ (a unit time-like) called the characteristic vector field of the manifold. Then we have $g(\xi, \xi) = -1$. (1.1)

Since $\xi$ is a unit concircular vector field, there exists a non-zero 1-form $\eta$ such that for $g(X, \xi) = \eta(X)$, (1.2)

the equation of the following form holds $(\nabla_X \eta)(Y) = \alpha [g(X, Y) \eta(Y) + \eta(X)\eta(Y)]$ (1.3)

for all vector fields $X$ and $Y$. Here $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$ and $\alpha$ is a non zero scalar function satisfying $(\nabla_X \alpha) = (\xi \alpha) - \alpha \eta(X)\xi - \rho \eta(Y)$. (1.4)

$\rho$ being a certain scalar function.

If we put $\phi_X = \frac{1}{\alpha} \nabla_X \xi$, (1.5)

then from (1.3) and (1.5) we have $\phi^2 X = X + \eta(X)\xi$, (1.6)

from which it follows that $\phi$ is a symmetric $(1,1)$ tensor called the structure tensor of the manifold. Bagewadi et al. [1,3,2] have
studied irrotational projective curvature tensor, quasi-conformal curvature tensor and \(D\)-conformal curvature tensor in \(K\)-contact, Kenmotsu and trans-Sasakian manifolds. Also the authors have proved that these manifolds are Einstein manifold.

The paper is organized as follows, section 1 and section 2 gives the brief introduction to \((LCS)_n\)-manifold, the basic equations of \((LCS)_n\)-manifold and definitions of \(\eta\)-Einstein and generalized \(\eta\)-Einstein manifolds. Section 3 deals with the study of irrotational conharmonic curvature tensor, where the Ricci tensor vanishes resulting (3.13), provided (3.12). Further, section 4, 5 and 6 are devoted to the study of irrotational concircular, \(M\)-projective and quasi-conformal curvature tensors respectively.

\((LCS)_n\)-manifold:

A differentiable manifold \(M\) of dimension \(n\) is called Lorentzian concircular structure manifold [briefly \((LCS)_n\)-manifold] if it admits a \((1,1)\) tensor field \(\phi\), a contravariant vector field \(\xi\), a covariant vector field \(\eta\) and a Lorentzian metric \(g\) which satisfies

\[
\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \quad (2.1)
\]

\[
\phi^2 X = \xi + \eta(X)\xi, \quad (2.2)
\]

\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)
\]

\[
\phi\xi = 0, \quad \eta(\phi X) = 0, \quad (2.4)
\]

for all \(X, Y \in TM\). Also in a \((LCS)_n\)-manifold \(M^n\), the following relations are satisfied [12]

\[
\eta(B(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (2.5)
\]

\[
B(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \quad (2.6)
\]

\[
B(X, \xi)Z = (\alpha^2 - \rho)[\eta(Z)X - g(X, Z)\xi], \quad (2.7)
\]

\[
B(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \quad (2.8)
\]

\[
B(\xi, \xi)\xi = (\alpha^2 - \rho)[X + \eta(X)\xi], \quad (2.9)
\]

\[
S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \quad (2.10)
\]

\[
Q\xi = (\alpha^2 - \rho)\xi, \quad (2.11)
\]

\[
(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \quad (2.11)
\]

\[
S(\phi X, \phi Y) = S(X, Y) + (n - 1)(\alpha^2 - \rho)\eta(X)\eta(Y), \quad (2.12)
\]

where \(R\), \(S\) and \(Q\) are the Riemannian curvature tensor, the Ricci curvature and the Ricci operator respectively.

A \((LCS)_n\)-manifold \(M^n\) is said to be an \(\eta\)-Einstein manifold if it satisfies

\[
S(U, V) = \alpha g(U, V) + \beta \eta(U)\eta(V), \quad (2.13)
\]

for any vector fields \(U\) and \(V\), where \(\alpha\) and \(\beta\) are smooth functions on \((M^n, g)\). If \(\beta = 0\) then \(\eta\)-Einstein manifold becomes Einstein manifold.

Next, from (2.13), we have

\[
Q U = \alpha U + \beta \eta(U)\xi, \quad (2.14)
\]

where \(Q\) is the Ricci operator.

Again, contracting (2.14) with respect to \(U\) and using (2.1), we see that

\[
r = n\alpha - \beta. \quad (2.15)
\]

Now, substituting \(X = \xi\) and \(Y = \xi\) in (2.13) and making use of (2.1) and (2.10), we obtain

\[
-\alpha + \beta = -(n - 1)(\alpha^2 - \rho). \quad (2.16)
\]

Equating (2.15) and (2.16), we get

\[
\alpha = \frac{r - n(\alpha^2 - \rho)}{n - 1}, \quad (2.17)
\]

and

\[
\beta = \frac{-n(n - 1)(\alpha^2 - \rho)}{n - 1}. \quad (2.18)
\]

A \((LCS)_n\)-manifold \(M^n\) is said to be a generalized \(\eta\)-Einstein manifold [16] if the following condition holds

\[
S(X, Y) = \lambda g(X, Y) + \mu \eta(X)\eta(Y) + \nu \Omega(X, Y), \quad (2.19)
\]

for any vector fields \(X\) and \(Y\), where \(\lambda, \mu\) and \(\nu\) are smooth functions and \(\Omega(X, Y) = g(\phi X, Y)\).

Irrotational Conharmonic curvature tensor

**Definition:** The conharmonic curvature tensor [4] on \((LCS)_n\)-manifold \(M\) of dimension \(n\) is defined as

\[
\Omega(X, Y) = \frac{1}{n}g(R(X, Y)Z - \frac{1}{n^2}g(Y, Z)X - g(X, Z)Y - g(Y, Z)X - g(X, Y)Z, \quad (3.1)
\]

for any vector fields \(X, Y\) and \(Z\) on \(M\).

**Definition:** Let \(D\) be a Riemannian connection, then the rotation (Curl) of
conharmonic curvature tensor $N$ on a Riemannian manifold $\mathcal{M}^n$ is defined as
\[ \text{Rot}N = (D_2N)(X,Y)Z + (D_3N)(U,Y)Z + (D_1N)(X,U)Z - (D_2N)(X,Y)U. \] (3.2)
With the help of second Bianchi identity, we have
\[ (D_2N)(X,Y)Z + (D_3N)(U,Y)Z + (D_1N)(X,U)Z = 0. \] (3.3)
In view of (3.3), (3.2) becomes
\[ \text{Rot}Z = -(D_2N)(X,Y)U. \] (3.4)
If the conharmonic curvature tensor is irrotational, then $\text{curl} N = 0$ and so by (3.4), we see that
\[ (D_2N)(X,Y)U = 0, \] (3.5)
which can be expressed as
\[ D_2(N(X,Y)U) = N(D_2X,Y)U + N(X,D_2Y)U + N(X,Y)D_2U. \] (3.6)
By replacing $\mathbf{u}$ by $\xi$ in (3.6), we get
\[ D_2(N(X,Y)\xi) = N(D_2X,Y)\xi + N(X,D_2Y)\xi + N(X,Y)D_2\xi. \] (3.7)
In (3.1), if we put $Z = \xi$ and using (2.6), (2.10) and (2.14), we have
\[ N(X,Y)\xi = \gamma(\eta(Y)X - \eta(X)Y). \] (3.8)
Where,
\[ \gamma = -\frac{\alpha^2 - \rho}{n-2}. \] (3.9)
Applying (3.8) in (3.7) and using (1.3), we obtain
\[ N(X,Y)\xi = \gamma\{\eta(Y)X - \eta(X)Y\}. \] (3.10)
Substituting $Z$ by $\phi Z$ in (3.10) and using (2.2), (2.4), one can get
\[ N(X,Y)\phi Z = \gamma\{\eta(Y)X - \eta(X)Y\}. \] (3.11)
Now, comparing (3.1) and (3.11), we see that
\[ \gamma\{\eta(Y)X - \eta(X)Y\}. \] (3.12)
On contracting with respect to $Y$ in (3.12) and making use of (3.9), we finally obtain
\[ \frac{(n-2)(\alpha^2 - \rho)}{n-2} g(Y,Z) + \frac{(n-2)}{n-2} \eta(Y)\eta(Z) - \frac{(n-2)}{n-2} g(\phi Z,Y) = 0. \] (3.13)
Thus, we can state the following:

**Theorem:** If an $n$-dimensional $(\mathcal{C})^n$ manifold satisfies irrotational conharmonic curvature tensor, then the Ricci tensor vanishes resulting (3.13), provided $(\alpha^2 - \rho) \neq 0$.

**Irrotational Concircular curvature tensor**

An interesting invariant of a concircular transformation is the concircular curvature tensor $Z$ and is given by [17, 18]
\[ Z = R - \frac{r}{n(n-1)} R^2. \] (4.1)
Where $R = g(Y,X)W - g(W,X)Y$. Here $R$ and $r$ denotes Riemannian curvature and scalar curvature respectively.

**Definition:** Let $\mathcal{D}$ be a Riemannian connection, then the rotation (Curl) of concircular curvature tensor $Z$ on a Riemannian manifold $\mathcal{M}^n$ is defined as
\[ \text{Rot}Z = (D_2Z)(W,X)Y + (D_3Z)(W,Y)X + (D_1Z)(W,Y)V - (D_2Z)(W,X)V. \] (4.2)
By virtue of second Bianchi identity, we have
\[ (D_2Z)(W,X)Y + (D_3Z)(W,Y)X + (D_1Z)(W,Y)V = 0. \] (4.3)
From (4.3), (4.2) reduces to
\[ \text{Rot}Z = -(D_2Z)(W,X)V. \] (4.4)
If the concircular curvature tensor is irrotational, then $\text{curl} Z = 0$ and by (4.4), we get
\[ (D_2Z)(W,X)V = 0. \] (4.5)
that can be written as
\[ D_2(Z(W,X)V) = 2D_1(W,X)V + Z(W,D_2V) + Z(W,D_2X)V + 2D_2(W,X)V. \] (4.6)
By treating $V = \xi$ in (4.6), we have
\[ D_2(Z(W,X)\xi) = 2(D_1W,X)\xi + Z(W,D_2V)\xi + Z(W,D_2X)\xi. \] (4.7)
In (4.1) if we put $Y = \xi$ and using (2.1) and (2.6), we obtain
\[ D_2(Z(W,X)\xi) = \sigma [g(Y,X)W - g(W,X)Y]. \] (4.8)
where
\[ \sigma = \left[ (\alpha^2 - \rho) \right] - \frac{n}{n(n-1)} \right]. \] (4.9)
By the use of (4.8) in (4.7) and using (1.3), we obtain
\[ D_2(Z(W,X)\xi) = \sigma [g(Y,X)W - g(W,X)Y]. \] (4.10)
On substituting $Y$ by $\phi Y$ and using (2.2), (2.4) and (4.8) in (4.10), we get...
Comparing (4.1) and (4.11) leads to

\[ Z(W,X)Y = d_2(g(Y,Z)W) - g(g(Y,W)X) - n(Y)\eta(X)W + n(Y)\eta(W)X \]  
(4.11)

Comparing (4.1) and (4.11) leads to

\[ d_2([g(Y,W)X] - g(g(Y,W)X) - n(Y)\eta(X)W + n(Y)\eta(W)X = R(W,X)Y - \frac{r}{n(n-2)} [g(Y,W)X - g(X,W)Y] \]  
(4.12)

Contracting (4.12) with respect to \( W \) and using (4.9), one can get

\[ S(X,Y) = \lambda_1 g(X,Y) + \mu_1 \eta(X)\eta(Y) + \nu_1 \phi(X,Y), \]  
(4.13)

where

\[ \lambda_1 = \frac{-(n-1)(\alpha^2 - \rho)}{n}, \]  
(4.14)

\[ \mu_1 = -\frac{[-(n-1)(\alpha^2 - \rho)]}{n}, \]  
(4.15)

\[ \nu_1 = \frac{-(n-1)^2(\alpha^2 - \rho)}{n}. \]  
(4.16)

Thus, we can state

**Theorem:** Let \( M^n \) be a \((LCS)_g\)-manifold in which concircular curvature tensor is irrotational then the manifold is generalized \( \eta \)-Einstein manifold.

**Irrotational \( M \)-projective curvature tensor**

In 1971, the authors [8] defined a tensor field \( W^n \) on a Riemannian manifold as follows

\[ W^n(X,Y,Z) = R(X,Y) - \frac{1}{2\eta^n} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)Q] \]  
(5.1)

where \( R, S \) denotes respectively Riemannian curvature, Ricci tensor and \( Q \) is the Ricci operator defined by

\[ S(X,Y) = g(\varphi(X,Y), Z). \]

**Definition:** Let \( D \) be a Riemannian connection, then the rotation (Curl) of \( M \)-projective curvature tensor \( Z \) on a Riemannian manifold \( M^n \) is defined as

\[ \text{Rot} Z = (D_2 W^n)(X,Y)U \]  
(5.2)

By virtue of second Bianchi identity, we have

\[ (D_2 W^n)(X,Y)Z + (D_2 W^n)(U,Y)Z + (D_2 W^n)(X,U)Z = 0. \]  
(5.3)

In view of (5.3), (5.2) becomes

\[ \text{Rot} Z = -(D_2 W^n)(X,Y)U. \]  
(5.4)

If the \( M \)-projective curvature tensor is irrotational, then \( \text{curl} W^n = 0 \) and by using (5.4), we see that

\[ (D_2 W^n)(X,Y)U = 0, \]  
(5.5)

which gives

\[ D_2 W^n(X,Y)U = W^n(X,D_2 Y)U + W^n(D_2 Y,X)U + W^n(X,Y)D_2 U \]  
(5.6)

Taking \( U = \xi \) in (5.6), we get

\[ D_2 W^n(X,Y)\xi = W^n(X,D_2 Y)\xi + W^n(D_2 Y,X)\xi + W^n(X,Y)D_2 \xi. \]  
(5.7)

Treating \( Z \) by \( \xi \) and using (2.6), (2.10) and (2.14) in (5.1), we obtain

\[ W^n(X,Y)\xi = \tau(\eta(X)\eta(Y) - \eta(X)\eta(Y)), \]  
(5.8)

where

\[ \tau = \frac{1}{2\eta^n} [(\alpha^n - \rho) + \frac{1}{n-1} (\alpha^n - \rho)]. \]  
(5.9)

By applying (5.8) in (5.7) and using (1.3), we get

\[ W^n(X,Y) \phi \xi = \tau(\eta(X)\eta(Y) - \eta(X)\eta(Y)). \]  
(5.10)

Substituting \( Z \) by \( \phi \xi \) and making use of (2.2), (2.4) and (5.8), the above equation yields

\[ W^n(X,Y)\phi \xi = \tau(\phi(X,Y) - \phi(X,Y) - \eta(X)\eta(Y) + \eta(Y)\eta(X)). \]  
(5.11)

Equating (5.1) and (5.11), we get

\[ \tau(\eta(X)\eta(Y) - \eta(X)\eta(Y)) = \tau(X,Y)Z - \frac{1}{2\eta^n} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)Q] - g(X,W)Y - g(X,W)Y. \]  
(5.12)

On contracting, (5.12) yields

\[ S(Y,Z) = \lambda_2 \eta(Y,\xi) + \mu_2 \eta(X)\eta(Y) + \nu_2 \phi(X,Y), \]  
(5.13)

where

\[ \lambda_2 = \frac{(n-1)(\alpha^n - \rho)}{n}, \]  
\[ \mu_2 = -\frac{(n-1)^2(\alpha^n - \rho)}{n}, \]  
and
\[ \nu_2 = \frac{(n-1)^2(\alpha^n - \rho)}{n}. \]

Thus, we can state the theorem:

**Theorem:** If the \( M \)-projective curvature tensor on a \((LCS)_g\)-manifold \( M^n \) is irrotational then the manifold is generalized \( \eta \)-Einstein manifold.
Irrotational Quasi-conformal curvature tensor

In 1968, Yano and Sawaki [19] defined and studied the concept of quasi-conformal curvature tensor \( \mathcal{C} \). According to the authors, quasi-conformal curvature tensor is given by

\[
\mathcal{C}(X,Y)\xi = \psi[(\eta(Y)X - \eta(X)Y)]
\]  

where \( \psi = (q_1 + q_2(n-1))\xi^2 - \frac{(a^2 - \rho)}{n} + \frac{q_2(n^2 - 2)^2}{n^2} \psi
\]

(6.9)

Now, applying (6.8) in (6.7) and by the use of (1.3), we see that

\[
\mathcal{C}(X,Y)\phi = \psi[(\eta(Y)X - \eta(X)Y)]
\]  

(6.10)

By replacing \( \xi \) by \( \phi \) in the above equation gives

\[
\mathcal{C}(X,Y)\phi = \psi[(\eta(Y)X - \eta(X)Y)]
\]  

(6.11)

Comparing (6.1) and (6.11), we have

\[
\psi[(\eta(Y)X - \eta(X)Y)] = \psi[(\eta(Y)X - \eta(X)Y)]
\]  

(6.12)

reference to (6.1) and (6.11), we have

\[
\mathcal{C}(X,Y)\phi = \psi[(\eta(Y)X - \eta(X)Y)]
\]

(6.13)

Therefore, we can state:

**Theorem:** If \( \text{a LCS}^n \)-manifold \( M^n \) in which quasi-conformal curvature tensor is irrotational then the manifold is generalized \( \eta \)-Einstein manifold.

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