Lorentzian α-Sasakian Manifold Satisfying Certain Pseudosymmetric Properties

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Abstract: The purpose of the present paper is to study Lorentzian α-Sasakian manifold satisfying pseudo Ricci symmetric, Ricci generalized pseudo symmetric and generalized pseudo-Ricci symmetric conditions. Finally, we prove that Lorentzian α-Sasakian manifold satisfying the condition S · R=0 reduces to Einstien manifold with scalar curvature $-\alpha^2$n(n-1).

Key words: Lorentzian α-Sasakian manifold, pseudo Ricci symmetric, Ricci generalized pseudo symmetric and generalized pseudo-Ricci symmetric, Ricci semisymmetric.

AMS Subject Classification: 53B30, 53C25, 53C50, 53D10.

1. Introduction:

Among the geometric properties of manifolds, symmetry is an important one and plays a significant role. Semisymmetric Riemannian manifolds was first studied by Cartan [1]. A Riemannian manifold M^n is called locally symmetric [12] if its curvature tensor R is parallel, i.e., $\nabla R = 0$. A Riemannian manifold M^n is Ricci-symmetric if its Ricci tensor S of type (0,2) satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. ARiemannian manifold M^n is said to be semisymmetric if its curvature tensor R satisfies $R(X,Y).R = 0$, $X,Y \in T(M^n)$, where $R(X,Y)$ acts on R as a derivation [11,6].

Over the last five decades, the concept of Ricci-symmetric manifolds has been weakened by several authors to a different extent such as Ricci-recurrent manifolds [9], Ricci semisymmetric manifolds [11], pseudo Ricci-symmetric manifolds [4] Ricci pseudo symmetric manifold [4,5].

Lorentzian manifold plays a pivotal role in differential geometric point of view because of its wide significance properties. An n-dimensional smooth differentiable manifold M with Lorentzian metric g is known as Lorentzian manifold. The idea of Lorentzian manifolds was first introduced by Matsumoto [7] in 1989. The same idea was independently studied by Mihai and Rosca [8]. A differentiable manifold M of dimension n is said to be a Lorentzian α-Sasakian manifold if it admits a (1,1)-tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g which satisfy the conditions

$$\phi^2 = I + \eta\otimes\xi, \quad (1.1)$$
$$\eta(\xi) = -1,\ \phi\xi = 0,\quad \eta(\phi X) = 0, \quad (1.2)$$
$$g(X,\xi) = \eta(X), \quad (1.3)$$
$$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad (1.4)$$
$$(\nabla_X\phi)(Y) = \alpha[g(X,Y)\xi + \eta(Y)X], \quad (1.5)$$

for all $X,Y \in T(M^n)$, where $T(M^n)$ is the Lie algebra of smooth vector fields on $T(M^n)$, α is smooth function on M^n and ∇ denotes the covariant differentiation operator of Lorentzian metric g [10,18].

On a Lorentzian α-Sasakian manifold [10,18], it can be shown that

$$\nabla_X\xi = \alpha\phi X, \quad (1.6)$$
$$(\nabla_X\eta)Y = \alpha g(\phi X,Y), \quad (1.7)$$

An extensive studies on Lorentzian α-Sasakian manifolds are seen in the following papers [10,16,17,18] and others.
Motivated by the above studies, we plan to study pseudosymmetric conditions on Lorentzian α-Sasakian manifold.

Our paper is structured in the following way: Section 2 contains basics of Lorentzian α-Sasakian manifold. Section 3 deals with the study of pseudo Ricci-symmetric Lorentzian α-Sasakian manifold. We proved that if an n-dimensional Lorentzian α-Sasakian manifold $(n > 1)$ is generalized pseudo Ricci-symmetric then the sum of 1-forms is always non-zero, in section 4. In Section 5, we have shown that if an n-dimensional Lorentzian α-Sasakian manifold $(n > 1)$ satisfies Ricci generalized pseudosymmetric condition then it is an Einstein manifold, provided $nf \neq 1$. In the last section, we study Lorentzian α-Sasakian manifold satisfying the condition S · R=0.

2. Lorentzian α-Sasakian manifolds

The authors [15,18] studied the characteristics of Lorentzian α-Sasakian manifold under different classes. The following relations hold on this manifold:

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \quad (2.1)$$
$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X], \quad (2.2)$$
$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \quad (2.3)$$
$$R(\xi,X)\xi = \alpha^2[X + \eta(X)\xi], \quad (2.4)$$
$$S(X,\xi) = (n-1)\alpha^2\eta(X), \quad (2.5)$$
$$S(\xi,\xi) = (n-1)\alpha^2, \quad (2.6)$$
$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\alpha^2\eta(X)\eta(Y), \quad (2.7)$$

where R and S are the curvature tensor and the Ricci tensor respectively.

An n-dimensional Lorentzian α-Sasakian manifold is said to be Einstein manifold if it satisfies

$$S(Y,Z) = \alpha g(Y,Z), \quad (2.8)$$

where α is a constant.

3. On Pseudo Ricci-symmetric Lorentzian α-Sasakian manifold

In 1988, Chaki introduced the notion of pseudo Ricci-symmetric $(PRS)_n$ manifolds [4]. The same concept was studied by Tarafdar [13] on different manifold.

Definition: A non-flat Riemannian manifolds (M^n, g) is called pseudo Ricci-symmetric if its Ricci tensor S is not identically zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) = 2G(X)S(Y,Z) + G(Y)S(X,Z) + G(Z)S(Y,X), \quad (3.1)$$

where G is a non-zero 1-form and

$$g(X,P) = G(X). \quad (3.2)$$

Suppose that equations (3.1) and (3.2) are satisfied in an n-dimensional Lorentzian α-Sasakian manifold $(n > 1)$. By taking cyclic sum of (3.1), one can get

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 4\{G(X)S(Y,Z) + G(Y)S(X,Z) + G(Z)S(Y,X)\}. \quad (3.3)$$

Now, admitting a cyclic Ricci tensor in (3.3), we have

$$G(X)S(Y,Z) + G(Y)S(X,Z) + G(Z)S(Y,X) = 0. \quad (3.4)$$

Plugging Z by ξ in (3.4) and using (2.5), we obtain

$$(n-1)\alpha^2G(X)\eta(Y) + (n-1)\alpha^2G(Y)\eta(X) + G(\xi)S(X,Y) = 0. \quad (3.5)$$

In a similar manner, treating $Y = \xi$ in the above equation and using (1.2) and (2.5), we get

$$(n-1)\alpha^2G(X) + 2(n-1)\alpha^2G(\xi)\eta(X) = 0, \quad (3.6)$$

which implies that

$$G(X) = 2G(\xi)\eta(X). \quad (3.7)$$

Again, continuing the process by putting $X = \xi$ in (3.6) and using (1.2), finally we see that

$$\eta(P) = 0. \qquad (3.8)$$

Hence we can state the following theorem:

Theorem: If a pseudo Ricci-symmetric Lorentzian α-Sasakian manifold $(n > 1)$ admits a cyclic parallel Ricci tensor then the 1-form G must vanish.

4. On Generalized Pseudo Ricci-symmetric Lorentzian α-Sasakian manifold

Definition: A non-flat Riemannian manifold (M^n, g) $(n > 1)$ is called generalized pseudo Ricci-symmetric [2] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(X,Y), \qquad (4.1)$$

where $A, B\ and\ C$ are three non-zero 1-forms.

Definition: Let M^n be a n-dimensional generalized pseudo Ricci-symmetric Lorentzian α-Sasakian manifold.

By substituting Z by ξ in (4.1) and using (2.5), we get

$$(\nabla_X S)(Y,\xi) = 2(n-1)\alpha^2 A(X)\eta(Y) + (n-1)\alpha^2 B(Y)\eta(X) + C(\xi)S(X,Y). \quad (4.2)$$

Also we find,

$$(\nabla_X S)(Y,\xi) = (n-1)\alpha^3 g(\phi\, X, Y) - \alpha S(\phi\, X, Y). \qquad (4.3)$$

On equating the RHS of (4.2) and (4.3), we obtain

$$(n-1)\alpha^3 g(\phi X, Y) - \alpha S(\phi X, Y) = 2(n-1)\ \alpha^2 A(X)\eta(Y) + (n-1)\alpha^2 B(Y)\eta(X) + C(\xi)S(X,Y). \qquad (4.4)$$

Now, treating $X = Y = \xi$ in (4.4) and using (1.2) and (2.6), we have

$$[2A(\xi) + B(\xi) + C(\xi)] = 0. \qquad (4.5)$$

Suppose $(n-1)\alpha^2 \neq 0$ then the 1-form $2A + B + C$ over the killing vector field ξ of (M^n, g) vanishes.

Further, replacing X by ξ in (4.2) and using (1.2), (2.5), (4.3) and (4.5), we get

$$B(Y) = -B(\xi)\eta(Y). \qquad (4.6)$$

Similarly, in (4.2) taking $Y = \xi$ and using (1.2), (2.5), (4.3) and (4.5), we have

$$2A(X) = -2A(\xi)\eta(X). \qquad (4.7)$$

Likewise, substituting $X = Y = \xi$ in (4.1) and using (2.5), (4.3) and (4.5), we see that

$$C(Z) = -C(\xi)\eta(Z). \qquad (4.8)$$

In view of (4.6), (4.7) and (4.8), finally we obtain

$$[2A(X) + B(X) + C(X)] = 0. \qquad (4.9)$$

Hence, we state the following theorem:
Theorem: If an n-dimensional Lorentzian α-Sasakian manifold $(n > 1)$ is generalized pseudo Ricci-symmetric then the sum of 1-forms is always non-zero.

5. On Ricci Generalized Pseudosymmetric Lorentzian α-Sasakian manifold

Definition: A Riemannian manifold (M^n, g) $(n > 1)$ is said to be Ricci generalized pseudosymmetric [14,16] if and only if the relation

$$R \cdot R = fQ(S,R), \qquad (5.1)$$

holds on the set $U = \{x \in M^n :\ Q(S,R) \neq 0\}$ at x, where f is a some function on U.
Also, we define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad (5.2)$$

and

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \quad (5.3)$$

respectively, where $X, Y, Z \in T(M^n)$, $T(M^n)$ is the set of all differentiable vector fields on M^n, A is the symmetric (0,2)-tensor.

Let us consider that an n-dimensional Ricci generalized pseudosymmetric Lorentzian α-Sasakian manifold $(n > 1)$.

In view of (5.1), we can have

$(R(X,Y) \cdot R)(U,V)W = f((X \wedge_S Y) \cdot R)(U,V)W,$ (5.4)

which follows that

$R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W = f[(X \wedge_S Y)R(U,V)W - R((X \wedge_S Y)U,V - R(U,(X \wedge_S Y)V)W - R(U,V)(X \wedge_S Y)W].$ (5.5)

From the equations (5.3) and (5.5), we get $R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W = f[S(Y,R(U,V)W)X - S(X,R(U,V)W)Y - S(Y,U)R(X,V)W + S(X,U)R(Y,V)W - S(Y,V)R(U,X) + S(X,V)R(U,Y)W - S(Y,W)R(U,V)X + S(X,W)R(U,V)Y].$ (5.6)

Replacing X and U by ξ in the above equation and after simplification, one can obtain
$\alpha^4 a g(V,W)Y - \alpha^2 R(Y,V)W - \alpha^4 g(Y,W)V = f[-\alpha^2 S(Y,V)\eta(W)\xi + \alpha^4 g(V,W)Y - (n-1)\,\alpha^2\, R(Y,V)W + (n-1)\,\alpha^4 g(Y,W)\,\eta(V)\,\xi - \alpha^2 S(Y,W)\eta(V)\xi - \alpha^2 S(Y,W)V + (n-1)\,\alpha^4 g(V,Y)\eta(W)\xi].$ (5.7)
Now, taking inner product with T in (5.7) and using (1.3), we get
$\alpha^4 g(V,W)g(Y,T) - \alpha^2 g(R(Y,V)W,T) - \alpha^4 g(Y,W)g(V,T = f[-\alpha^2 S(Y,V)\eta(W)\eta(T) + (n-1)\alpha^4 g(V,W)g(Y,T) - (n-1)\alpha^2 g(R(Y,V)W,T) + (n-1)\alpha^4 g(Y,W)\eta(V)\eta(T) - \alpha^2 S(Y,W)\eta(V)\eta(T) - \alpha^2 S(Y,W)g(V,T) + (n-1)\alpha^4 g(V,Y)\eta(W)\eta(T)].$ (5.8)

Contracting the above equation, we obtain
$[S(Y,T) - (n-1)\alpha^2 g(Y,T)] = nf[S(Y,T) - (n-1)\alpha^2 g(Y,T)],$ (5.9)
which implies
$(1-nf)[S(Y,T) - \alpha^2(n-1)g(Y,T)] = 0.$ (5.10)
Then either $f = \frac{1}{n}$ or the manifold is an Einstein manifold of the following form
$S(Y,T) = (n-1)\,\alpha^2\, g(Y,T).$ (5.11)
Therefore, the above equation yields
$r = n(n-1)\,\alpha^2,$ (5.12)
Where r is a scalar curvature.
Thus, the theorem can be stated as follows:
Theorem: Let (M^n) be an n-dimensional Lorentzian α-Sasakian manifold $(n > 1)$. If such a manifold satisfies Ricci generalized pseudosymmetric condition then the manifold is an Einstein manifold, provided $nf \neq 1$ with scalar curvature as in (5.12).

6. Lorentzian α-Sasakian manifold satisfying the condition $S \cdot R = 0$.

Consider an n-dimensional Lorentzian α-Sasakian manifold $(n > 1)$ satisfying the curvature condition $S \cdot R = 0$. Then, we have
$(S(X,Y) \cdot R)(U,V)W = 0,$ (6.1)
which implies
$(X \wedge_S Y)R(U,V)W + R\big((X \wedge_S Y)U,V\big)W + R(U,(X \wedge_Y)V) + R(U,V)(X \wedge_S Y)W = 0.$ (6.2)
Making use of (5.3) in (6.2), we get

$$S(Y,R(U,V)W)X - S(X,R(U,V)W)Y + S(Y,U)R(X,V)W - S(X,U)R(Y,V)W + S(Y,V)R(U,X)W - S(X,V)R(U,Y)W + S(Y,W)R(U,V)X - S(X,W)R(U,V)Y = 0. \quad (6.3)$$

Substituting $U = W = \xi$ in (6.3) and using (2.3), (2.4) and (2.5), we find that

$\alpha^2 S(Y,V)X - \alpha^2 S(X,V)Y + \alpha^2 S(Y,V)X - \alpha^2 S(X,V) + 2(n-1)\alpha^4\eta(V)\,\eta(Y)X - 2(n-1)\alpha^4\eta(V)\eta(X)Y - 2(n-1)\alpha^4\eta(X)\eta(Y) + \alpha^2 S(Y,V)\,\eta(X)\xi - \alpha^2 S(X,V)\eta(Y)\xi + -(n-1)\alpha^4 g(V,Y)\eta(X)\xi.$ (6.4)
By taking innerproduct of the above equation with ξ and replacing X by ξ, we obtain

$\alpha^2[S(U,W)+(n-1)\alpha^2 g(U,W)]=0,$ (6.5)

Which means either $\alpha^2=0$ (contradiction) or the manifold becomes Einstein manifold. So, we have

$S(U,W)=-(n-1)\alpha^2 g(U,W),$ (6.6)

Which in turn yields

$r=-n(n-1)\alpha^2.$ (6.7)

This leads to the following statement:

Theorem: If an n-dimensional Lorentzian α-Sasakian manifold $(n>1)$ satisfies the curvature condition $S\cdot R=0$, then the manifold is an Einstein manifold with negative scalar curvature as in (6.7).

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